1. Let S_{17} be the permutation group on $\{1, 2, 3, \ldots, 17\}$ and let $\sigma = (2\ 5\ 3\ 6)(11\ 3\ 6\ 5\ 7)(7\ 11\ 5\ 8\ 3)(1\ 4) \in S_{17}$.

   (a) [**1 point**] Write σ as a product of 2-cycles.

   > Solution: One can write
   >
   > $$\sigma = (2\ 5)(5\ 3)(3\ 6)(11\ 3)(3\ 6)(6\ 5)(5\ 7)(7\ 11)(11\ 5)(5\ 8)(8\ 3)(1\ 4).$$
   >
   > **Any correct answer is 1 point.**

   (b) [**2 points**] Write the inverse of σ as a product of disjoint cycles.

   > Solution: We write this permutation as a product of disjoint cycles:
   >
   > $$\sigma = (3\ 11\ 7\ 6)(8\ 2\ 5)(1\ 4).$$
   >
   > Since disjoint cycles commute the inverse of σ is the product of the inverses of the cycles:
   >
   > $$\sigma^{-1} = (6\ 7\ 11\ 3)(8\ 5\ 2)(1\ 4).$$
   >
   > **Writing as a product of disjoint cycles is 1 point; finding the inverse is 1 point.**

   (c) [**2 points**] Compute the kernel of the homomorphism
   $$\varphi : \mathbb{Z} \to S_{17}$$
   $$k \mapsto \sigma^k.$$

   > Solution: By definition
   > $$\ker(\varphi) = \{k \in \mathbb{Z} \ : \ \sigma^k = (1)\}.$$
   >
   > This implies that $\mathrm{ord}(\sigma) = \mathrm{lcm}(4, 3, 2) = 12$ divides k. Therefore we have $\ker(\varphi) = 12\mathbb{Z}$.
   > **Arguing that $\mathrm{ord}(\sigma) \mid k$ is 0.5 points, computing $\mathrm{lcm}(4, 3, 2) = 12$ 0.5 points and concluding $\ker(\varphi) = 12\mathbb{Z}$ is 1 point.**

   (d) [**2 points**] Find a subgroup of S_{17} containing σ and isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

   > Solution: By CRT, we have $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong \mathbb{Z}/60\mathbb{Z}$. So we need to find a cyclic subgroup of order 60 which contains σ. The cyclic group generated by
   >
   > $$\tau = (3\ 11\ 7\ 6)(8\ 2\ 5)(1\ 4)(2\ 9\ 12\ 13\ 14)$$
   >
   > contains σ since $\tau^{25} = \sigma$. Moreover, since $\mathrm{lcm}(4, 3, 2, 5) = 60$, this subgroup consists of 60 elements.
   > **If they figure out that the group should be generated by τ and a 5-cycle then 1 points, and if they choose a correct 5-cycle (i.e., a disjoint one) then it is 2 points. They should also argue why these permutations generate a group isom. to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ (if they don't then deduct 0.5 points).**

2. Consider the group

$$G = \left\{ \begin{bmatrix} x & y \\ z & t \end{bmatrix} : x, y, z, t \in \mathbb{Z}/5\mathbb{Z},\ xt - zy \neq 0 \right\}$$

with respect to matrix multiplication. Let

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} : a, b \in \mathbb{Z}/5\mathbb{Z},\ a \neq 0 \right\}.$$

(Recall that $(\mathbb{Z}/p\mathbb{Z})^{\times}$, where p is a prime, is a group with respect to multiplication.)

(a) [**3 points**] Show that H is a subgroup of G but not a normal subgroup.

Solution: Since $a \neq 0$ we have $a \cdot 1 - 0 \cdot b \neq 0$, so $H \subset G$. [**0.5 points**]

- [**0.5 points**] $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$;

- [**0.5 points**] For every $\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} \in G$ we have

$$\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b+ad & ac \end{bmatrix} \in G.$$

- [**0.5 points**] For every $\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \in G$ the matrix $\begin{bmatrix} 1 & 0 \\ -ba^{-1} & a^{-1} \end{bmatrix} \in G$ (as $a \neq 0$ and every non zero element in $\mathbb{Z}/5\mathbb{Z}$ is invertible) such that

$$\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -ba^{-1} & a^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So H is a subgroup of G. By definition H is normal iff $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$ so to show that H is not normal it suffices to find $g \in G$ and $h \in H$ such that $ghg^{-1} \notin H$. Let

$$g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in G \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in H.$$

Then we get

$$ghg^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \notin H.$$

[**1 point**]

(b) [**5 points**] Find the Sylow p-groups in H. Which ones are normal in H?

Solution: Since $a, b \in \mathbb{Z}/5\mathbb{Z}$ and $a \neq 0$ we have $|H| = 20 = 2^2 \cdot 5$. The possible Sylow p-subgroups are Sylow 2-groups of order 4 and Sylow 5-groups of order 5. We first determine n_2 and n_5. By Sylow theory we have

$$n_2 \mid 5 \quad \text{and} \quad n_2 \equiv 1 \mod 2$$

$$n_5 \mid 4 \quad \text{and} \quad n_5 \equiv 1 \mod 5.$$

Therefore we get $n_2 = 1$ or 5 and $n_5 = 1$. So there is only one Sylow 5-group, denote it by P_5. Since $|P_5| = 5$, it is cyclic so we need to find an element in H of order 5. We determine $a, b \in \mathbb{Z}/5\mathbb{Z}$ such that

$$\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix}^5 = \begin{bmatrix} 1 & 0 \\ a^4b+a^3b+a^2b+ab+b & a^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So one can take $a = 1$, and $b = 1$. So $P_5 = \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle$.

We will now determine the Sylow 2-groups. Let P_2 be a Sylow 2-group. Since $|P_2| = 4$, we have
$$P_2 \cong \mathbb{Z}/4\mathbb{Z} \quad \text{or} \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$
We have
$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ a^3b + a^2b + ab + b & a^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
Therefore, an element has order 4 iff $a^4 = 1 \mod 5$, $a^2 \neq 1 \mod 5$ and $a^3b + a^2b + ab + b = 0$ mod 5 iff $(a,b) \in \{(2,0),(2,1),(2,2),(2,3),(2,4)\}$. Therefore there are 5 Sylow 2-groups and they are isomorphic to $\mathbb{Z}/4\mathbb{Z}$, which are
$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} \right\rangle.$$

Computing n_2 and n_5 is 1 point; writing the Sylow 5-group is 1 point; showing that the Sylow 2-group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ is 1 point; writing the Sylow 2-groups is 1 point; determining the normal ones correctly is 1 point.

(c) **[3 points]** The group H acts on $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ via the rule
$$\left(h, \begin{bmatrix} x \\ y \end{bmatrix} \right) \mapsto h \begin{bmatrix} x \\ y \end{bmatrix}.$$

Is this action faithful; transitive; or fixed point free? Explain.

Solution:
- **[1 point]** The action is faithful: Let $\begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ d & c \end{bmatrix}$ be arbitrary elements in H. Then we have
$$\begin{bmatrix} x \\ bx + ay \end{bmatrix} \neq \begin{bmatrix} x \\ dx + cy \end{bmatrix} \iff bx + ay \neq dx + cy \mod 5 \iff (b-d)x \neq (a-c)y \mod 5.$$

  If $b \neq d$, one can take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and if $a \neq c$, one can take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This proves that the action is faithful.
- **[1 point]** The action is not transitive: take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then
$$H \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ a \end{bmatrix} \ : \ a \in \mathbb{Z}/5\mathbb{Z} \right\} \neq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.$$
- **[1 point]** The action is not fixed point free since it fixes $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(d) Write $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a union of disjoint orbits of the action in (c).

Solution: We have
$$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} = H\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cup H\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cup H\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cup H\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cup H\begin{bmatrix} 3 \\ 0 \end{bmatrix} \cup H\begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Complete answer is 2 points and any reasonable attempt with a mistake 1 or 1.5 points depending on the mistake.

3. Let G be a group with precisely one non-trivial proper normal subgroup N. Suppose $|N| = 3$.

(a) **[2 points]** If $f : G \to \mathbb{Z}/16\mathbb{Z}$ is a surjective homomorphism, what are the possible orders for the group G?

Solution: By the homomorphism theorem, we have

$$G/\ker(f) \cong \mathbb{Z}/16\mathbb{Z}$$

Since $\ker(f)$ is a normal subgroup in G, we have $\ker(f) = \{1\}, N$ or G. [**0.5 points**]
Suppose $\ker(f) = \{1\}$. Then by the homomorphism theorem we have $|G| = 16$ on the other
hand we have $3 \mid |G|$. So we get a contradiction. [**0.5 points**] Since f is a surjective map,
the kernel cannot be G. [**0.5 points**]
Therefore, we have $\ker(f) = N$ and hence we get $|G|/|N| = 16$ by the homomorphism theorem.
So $|G| = 48$. [**0.5 points**]

(b) [**2 points**] Show that there is no injective homomorphism from G to $\mathbb{Z}/16\mathbb{Z}$.

Solution: If there is an injective homomorphism, then the kernel is trivial. Hence the $|G| \mid 16$.
This is impossible since $3 \mid |G|$.
Correct answer is 2 points.

4. (a) [**4 points**] List all non-isomorphic abelian groups of order between 60 and 97 (both bounds
are included) with at least two elementary divisors.

Solution: We first list all abelian non-isomorphic groups with two elementary divisors, in
other words, we will list abelian groups in the form $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z}$. [**0.5 points**] These abelian
groups have order $a^2 b$ for positive integers a, b. So we list $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with $60 \le a^2 b \le 97$:

$$\{(2, b) : 15 \le b \le 24\}$$
$$\{(3, b) : 7 \le b \le 10\}$$
$$\{(4, b) : 4 \le b \le 6\}$$
$$\{(5, 3)\}$$
$$\{(8, 1), (9, 1)\}.$$

Complete list is 2 points, deduct points (proportionally) if the list is not complete.
Now assume that there are 3 elementary divisors, i.e, abelian groups are isomorphic to
$\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/ab\mathbb{Z} \times \mathbb{Z}/abc\mathbb{Z}$ so the cardinality of such groups are in the form $a^3 b^2 c$ for some
positive integers a, b, c. [**0.5 points**] Then we get

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}.$$

Same here, each group gives 0.25 points. (0.5 points in total)
If the number of divisors is larger than 3 then there are at least 4 positive integers a, b, c, d.
Similarly, we get $60 \le a^4 b^3 c^2 d \le 97$. When $b = c = d = 1$, we get the case $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \times$
$\mathbb{Z}/a\mathbb{Z}$ with $a = 3$. If any of b, c, d is greater than 1, there are no integers in the range. So we
conclude that the list above is the complete list. [**0.5 points**]

(b) [**3 points**] Find the elementary divisors of the multiplicative group $(\mathbb{Z}/88\mathbb{Z})^\times \times (\mathbb{Z}/63\mathbb{Z})^\times$.

Solution: By the CRT (Theorem II.3.4.) we have

$$(\mathbb{Z}/88\mathbb{Z})^\times \times (\mathbb{Z}/63\mathbb{Z})^\times \cong (\mathbb{Z}/8\mathbb{Z})^\times \times (\mathbb{Z}/11\mathbb{Z})^\times \times (\mathbb{Z}/7\mathbb{Z})^\times \times (\mathbb{Z}/9\mathbb{Z})^\times.$$

[0.5 points]
We have $(\mathbb{Z}/8\mathbb{Z})^\times = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ and since every non identity element in this abelian group has order 2 we have
$$(\mathbb{Z}/8\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

[0.5 points]
We have $(\mathbb{Z}/9\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$ and since the cardinality of this abelian group is 6 we have
$$(\mathbb{Z}/9\mathbb{Z})^\times \cong \mathbb{Z}/6\mathbb{Z}.$$

[0.5 points]
The cardinality of $(\mathbb{Z}/11\mathbb{Z})^\times$ is 10 and since it is an abelian group, it is isomorphic to $\mathbb{Z}/10\mathbb{Z}$. Similarly, the group $(\mathbb{Z}/7\mathbb{Z})^\times$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. **[0.5 points]** So we have

$$\begin{aligned}
(\mathbb{Z}/88\mathbb{Z})^\times \times (\mathbb{Z}/63\mathbb{Z})^\times &\cong (\mathbb{Z}/8\mathbb{Z})^\times \times (\mathbb{Z}/11\mathbb{Z})^\times \times (\mathbb{Z}/7\mathbb{Z})^\times \times (\mathbb{Z}/9\mathbb{Z})^\times \\
&\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \\
&\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \\
&\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}.
\end{aligned}$$

So the elementary divisors are $2, 2, 2, 6, 30$.**[1 point]**
If they immediately use $(\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$ then don't deduct any points. They should show the other two cases.

5. **True/False.** Prove the following statements if they are correct and disprove if they are wrong.

   (a) **[2 points]** The group $G = S_6 \times D_5$ contains a normal subgroup H such that G/H is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

   Solution: **False.** Assume that there is a normal subgroup H in G. Since $|G| = 2^5 \cdot 3^2 \cdot 5^2$, we have
   $$7 = |G/H| = \frac{|G|}{|H|} \in \{k : k \mid 2^5 \cdot 3^2 \cdot 5^2\}.$$
   Since 7 does not divide $|G|$, there cannot be a normal subgroup H such that G/H is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.
   Correct answer is 2 points

   (b) **[2 points]** $D_{25} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic $D_{10} \times \mathbb{Z}/5\mathbb{Z}$.

   Solution: **False.** The dihedral group D_{25} has an element of order 25, namely the rotation ρ_{25} by angle $2\pi/25$. Therefore, $(\rho_{25}, 0) \in D_{25} \times \mathbb{Z}/2\mathbb{Z}$ has order 25. On the other hand the order of the elements (a, b) in $D_{10} \times \mathbb{Z}/5\mathbb{Z}$ is $\text{lcm}(|a|, |b|) = \{1, 2, 5, 10\}$ so there is no element of order 9. Therefore these two groups cannot be isomorphic.
   Correct answer is 2 points

   (c) **[2 points]** Let G be a group of order 34 and let H, K be subgroups of G such that $H \subset K \subset G$ with $H \neq K$ and $K \neq G$. Then H is the trivial subgroup.

   Solution: **True.** By Lagrange's theorem, we have $|H| \mid |K| \mid |G|$ and since $|K| \neq |G|$, we have $|K| \in \{1, 2, 17\}$ but since 2 and 17 are primes and $|H| \neq |K|$, we have $|H| = 1$. So it follows.
   Correct answer is 2 points

GOOD LUCK! ☺